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# FREE VIBRATION OF EULER-BERNOULLI BEAMS MADE OF AXIALLY FUNCTIONALLY GRADED MATERIALS

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## INTRODUCTION

Functionally graded (FG) beams are composites characterized by the volume fraction of different materials which is varied continuously with the thickness and/or the length of the beam. Through an appropriate selection of the volume fraction the FG beam with expected thermal and mechanical properties can be obtained. Therefore, the FG beams can be used in various engineering applications.

In this study the solution to the free vibration problem of axially graded beams with a non-uniform cross-section has been presented. The proposed approach relies on replacing of the functions characterizing the functionally graded beams by piecewise exponential functions. The frequency equation has been derived for axially graded beams divided into an arbitrary number of subintervals. Numerical examples show the influence of the parameters of the functionally graded beams on the free vibration frequencies for different boundary conditions.

## FORMULATION OF THE PROBLEM

### The governing equation

$$\frac{d^2}{d\xi^2}\left[E(\xi)I(\xi)\frac{d^2W}{d\xi^2}\right]-L^4\omega^2\rho(\xi)A(\xi)W=0, \quad 0<\xi<1$$

$W(\xi)$ - the amplitude of vibration

$E(\xi)$ - the modulus of elasticity

$I(\xi)$ - the moment of inertia

$\rho(\xi)$ - the material density

$A(\xi)$ - the cross-section area

$\omega$ - the circular frequency of vibration

$L$ - the length of beam

### Basic assumptions

Assumption 1

$$E(\xi)I(\xi)=d_0 g(\xi) \qquad \rho(\xi)A(\xi)=m_0 h(\xi), \qquad 0<\xi<1$$

where

$$d_0=E(0)I(0),\; m_0=\rho(0)A(0)$$
$$g(\xi)\cong d_i e^{2\beta_i\xi}$$
$$h(\xi)\cong m_i e^{2\beta_i\xi} \qquad \xi_{i-1}<\xi<\xi_i, \qquad i=1,\dots,n$$

Assumption 2

$$g(\xi_{i-1})=d_i e^{2\beta_i\xi_{i-1}}$$
$$g(\xi_i)=d_i e^{2\beta_i\xi_i}$$
$$g(\xi_0)=1 \qquad\qquad i=1,\dots,n$$
$$h\left(\frac{\xi_i+\xi_{i-1}}{2}\right)=m_i e^{\beta_i(\xi_i+\xi_{i-1})}$$
$$h(\xi_0)=1$$

Hence

$$\beta_i=\frac{1}{2(\xi_i-\xi_{i-1})}\ln\frac{g(\xi_i)}{g(\xi_{i-1})}$$
$$d_i=g(\xi_i)e^{-2\beta_i\xi_i}$$
$$m_i=h\left(\frac{\xi_i+\xi_{i-1}}{2}\right)e^{-\beta_i(\xi_i+\xi_{i-1})}$$

## SOLUTION TO THE PROBLEM

### Denotations

$$W(\xi)=W_i(\xi) \qquad \xi_{i-1}<\xi<\xi_i$$
$$\Omega^2=\frac{m_0 m_1}{d_0 d_1}L^4\omega^2 \qquad i=1,\dots,n$$
$$\mu_i^2=\frac{m_i d_1}{m_1 d_i}$$

### The governing equation for i-th segment

$$\frac{d^2}{d\xi^2}\left[e^{2\beta_i\xi}\frac{d^2W_i}{d\xi^2}\right]-\Omega^2\mu_i^2 e^{2\beta_i\xi}W_i=0 \qquad \begin{matrix}\xi_{i-1}<\xi<\xi_i\\ i=1,\dots,n\end{matrix}$$

### Continuity conditions

$$W_i(\xi_i)=W_{i+1}(\xi_i) \qquad \frac{dW_i}{d\xi}(\xi_i)=\frac{dW_{i+1}}{d\xi}(\xi_i) \qquad i=1,\dots,n-1$$
$$\frac{d^2W_i}{d\xi^2}(\xi_i)=\frac{d^2W_{i+1}}{d\xi^2}(\xi_i) \qquad \frac{d^3W_i}{d\xi^3}(\xi_i)=\frac{d^3W_{i+1}}{d\xi^3}(\xi_i)$$

### Boundary conditions

Clamped-clamped beam (C-C)

$$W_1(0)=0,\; \frac{dW_1}{d\xi}(0)=0,\; W_n(1)=0,\; \frac{dW_n}{d\xi}(1)=0$$

Pinned-pinned beam (P-P)

$$W_1(0)=0,\; \frac{d^2W_1}{d\xi^2}(0)=0,\; W_n(1)=0,\; \frac{d^2W_n}{d\xi^2}(1)=0$$

Clamped-pinned beam (C-P)

$$W_1(0)=0,\; \frac{dW_1}{d\xi}(0)=0,\; W_n(1)=0,\; \frac{d^2W_n}{d\xi^2}(1)=0$$

Pinned-clamped beam (P-C)

$$W_1(0)=0,\; \frac{d^2W_1}{d\xi^2}(0)=0,\; W_n(1)=0,\; \frac{dW_n}{d\xi}(1)=0$$

Clamped-free beam (C-F)

$$W_1(0)=0,\; \frac{dW_1}{d\xi}(0)=0,\; \frac{d^2W_n}{d\xi^2}(1)=0,\; \frac{d}{d\xi}\left(e^{2\beta_i\xi}\frac{d^2W_n}{d\xi^2}\right)(1)=0$$

Free-clamped beam (F-C)

$$\frac{d^2W_1}{d\xi^2}(0)=0,\; \frac{d}{d\xi}\left(e^{2\beta_i\xi}\frac{d^2W_1}{d\xi^2}\right)(0)=0, W_n(1)=0,\; \frac{dW_n}{d\xi}(1)=0$$

### The general solutions

$$W_i(\xi)=e^{-\beta_i\xi}\left(A_i\cos\delta_i\xi+B_i\sin\delta_i\xi+C_i\cosh\bar{\delta}_i\xi+D_i\sinh\bar{\delta}_i\xi\right)$$
$$\xi_{i-1}<\xi<\xi_i, \qquad i=1,\dots,n$$

where

$$\delta_i=\sqrt{\mu_i\Omega-\beta_i^2},\quad \bar{\delta}_i=\sqrt{\mu_i\Omega+\beta_i^2},\; A_i,B_i,C_i,D_i\in R$$

### The system of linear equations

$$\mathbf{A}(\omega)\cdot\mathbf{X}=\mathbf{0}$$

where

$$\mathbf{X}=[A_1,B_1,C_1,D_1,\dots,A_n,B_n,C_n,D_n]^T$$
$$\mathbf{A}(\omega)=[a_{kj}]_{4n\times4n}$$

### The condition for non-trivial solution

$$\det\mathbf{A}(\omega)=0$$

## NUMERICAL RESULTS

Table 1 The first three non-dimensional free vibration frequencies for $g(\xi)=(1+\gamma\xi)^3$, $h(\xi)=1+\gamma\xi$, clamped-clamped beam

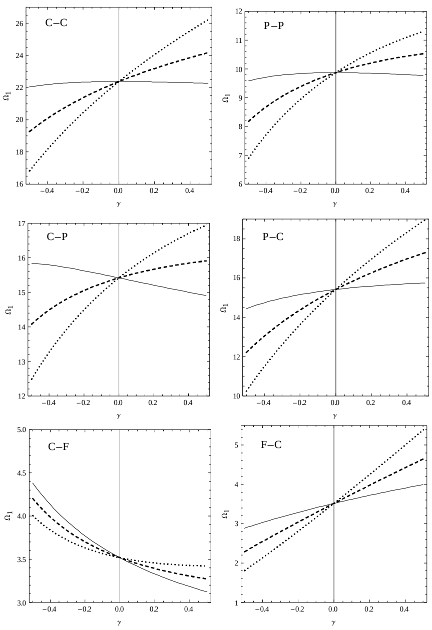
| γ | Power series method | Huang, Li (2010) | Present |
|---|---|---|---|
| -0.1 | 21.2409777868 | 21.24097778688 | 21.242905 |
| | 58.5500545739 | 58.55005461550 | 58.567526 |
| | 114.780241659 | 114.78027750905 | 114.824704 |
| 0 | 22.3732854478 | 22.37328544806 | 22.373285 |
| | 61.6728228676 | 61.67282294761 | 61.672823 |
| | 120.903391727 | 120.90340027002 | 120.903392 |
| 0.1 | 23.4796072481 | 23.47960724845 | 23.460013 |
| | 64.7210676329 | 64.72106768601 | 64.678046 |
| | 126.878016311 | 126.87805071630 | 126.802905 |
| 0.2 | 24.5634175322 | 24.5634175326 | 24.508817 |
| | 67.7047553171 | 67.7047553184 | 67.596273 |
| | 132.723976757 | 132.7240684027 | 132.546612 |

Table 2 The first three non-dimensional free vibration frequencies for $g(\xi)=(1+\gamma\xi)^3$, $h(\xi)=1+\gamma\xi$, clamped-pinned beam

| γ | Power series method | Huang, Li (2010) | Present |
|---|---|---|---|
| -0.1 | 14.8488960557 | 14.84889605539 | 14.844562 |
| | 47.6370371901 | 47.63703719174 | 47.647237 |
| | 99.171635183 | 99.17165323722 | 99.206918 |
| 0 | 15.4182057169 | 15.41820571698 | 15.418206 |
| | 49.964862032 | 49.96486203816 | 49.964862 |
| | 104.247696458 | 104.24770194514 | 104.247696 |
| 0.1 | 15.968709884 | 15.96870988416 | 15.950015 |
| | 52.2372268871 | 52.23722689317 | 52.198883 |
| | 109.202352455 | 109.20235370558 | 109.134912 |
| 0.2 | 16.5028988943 | 16.50289889399 | 16.445277 |
| | 54.4614625302 | 54.46146253076 | 54.360368 |
| | 114.051623344 | 114.05163085534 | 113.888586 |

Huang Y., Li X.-F., 2010, A new approach for free vibration of axially functionally graded beams with non-uniform cross-section, *Journal of Sound and Vibration*, **329**, 2291-2303

Fig.1 The first non-dimensional free vibration frequency as a function of γ for α=1 (solid line), α=2 (dashed line), α=3 (dotted line) for different boundary conditions, $g(\xi)=(1+\gamma\xi)^\alpha$, $h(\xi)=1+\gamma\xi$




## CONCLUSIONS

In this study a new method, which is capable of computing the free vibration frequencies of axially functionally graded beams is presented. An exact solution is derived for the axially piece-wise exponential graded beams with various combinations of clamped, pinned and free ends. A high agreement of the numerical results obtained by using the presented method with the results obtained by using the power series method, as well as with results given by other authors was also observed.